solid curves) and for the second normal wave for $k_0a = 10$ (the dashed curves). The normal wave damping coefficient in the presence of a permeable boundary considerably exceeds the transverse wave damping coefficient.

The influence of overflows through the interface of the media on normal wave propagation reveals the possibility of an experimental determination of the filtration characteristics of a medium by means of the parameters of these waves.

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ON THE STATE OF STRESS AND STRAIN OF LAYERED PLATES OF NON-SYMMETRIC CONSTRUCTION*

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An asymptotic analysis is performed of the elasticity theory problem of the deformation of a thin multilayer **anisotropic plate** in a three-dimensional formulation without assumption regarding the regularity of the plate construction and the nature of the layer or packet deformation as a whole.

Results /1/ are used of an investigation of the solutions of elliptic boundary value problems in thin domains. The relative packet height is the small parameter h. A system of equations is obtained for the limit problem (as $h \to 0$), effective plate stiffness characteristics are found, and specific examples of their analysis are presented.

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Asymptotic methods of constructing the solutions of problems of the theory of thin plates are developed in /1-8/; isotropic multilayer plates and anisotropic two-layer beams were studied /9-11/ by the method described in /3/.

An important feature of anisotropic laminar plates of non-symmetric construction is that the state of stress in any section parallel to the middle surface is characterized during their deformation by interaction of the bending-torsion and tension-shear states. The limit bending equations for a thin plate of complex structure are derived by using the Kirchhoff hypothesis in /5, 12/.

1. Formulation of the problem. A spatial problem is considered for the elasticity theory problem of the deformation of a thin cylindrical domain Ω whose side surface is clamped stiffly while loads of a given intensity are applied to the bases. The plate occupies the domain Ω and is a packet of anisotropic layers between which ideal mechanical contact is realized. The layers are arranged parallel to the plate bases.

A Cartesian coordinate system $\mathbf{x} = (x_1, x_2, x_3)$ is introduced in such a way that its origin O is in the domain $\Omega = \omega \times [-H, H]$ while the axis Ox_3 is normal to the packet middle surface ω. It is assumed that the ratio of the plate height $2H$ to the characteristic linear dimension D of the domain ω is a small parameter $h > 0$. Now, let the quantity D be reduced to one by a change of scale. Moreover, it is assumed that all the quantities with the dimensions of length are referred to D. The coordinate symbols utilized above and the notation used for the domains are conserved.

Mathematically the problem can be formulated as follows. The plate Ω consists of $m + n$ layers $\Omega^r = \{\mathbf{x} : \mathbf{x}' = (x_1, x_2) \in \omega, hl_r \leqslant x_3 \leqslant hl_{r+1}\}$ denoted by numbers in the range $[-m, n]$; $\Omega^0 = \varnothing$, $\sum\limits_{-m}^{n} l_r = 1$. The equilibrium equations

$$\sigma_{ij,j}^r (\mathbf{x}, h) = 0, \quad x \in \Omega^r; \quad i, j = 1, 2, 3 \tag{1.1}$$

are satisfied in each of the domains Ω^r.

The components σ_{ij}^r and ε_{ij}^r of the stress and strain tensors are connected by the Hooke's law. It is convenient to write this in a form analogous to that utilized in /5, 12/. Namely, if $\mathbf{e}$ and Σ are six-dimensional vectors with the components $\varepsilon_{11}, \varepsilon_{22}, \gamma_{12}, \gamma_{13}, \gamma_{23}, \varepsilon_{33}$ and $\sigma_{11}, \ldots, \sigma_{33}$, respectively, where γ_{ij} are the shear strain components, then

$$\Sigma_i^r (x, h) = B_{ij}^r e_j (x, h), \quad x \in \Omega^r; \quad \mathbf{B} = \| B_{jk} \|_{j,k=1}^{6} \tag{1.2}$$

and $\mathbf{B}$ is a symmetric matrix of the elastic constants whose elements are four 3×3 matrices: B_1 and B_3 on the principal diagonals, and also the matrices B_2 and B_2^T. The superscript T denotes the transpose.

Adjoint conditions

$$\begin{aligned} &\boldsymbol{\sigma}^{(3)} (\mathbf{x}', hl_r + 0; h) - \boldsymbol{\sigma}^{(3)} (\mathbf{x}', hl_r - 0; h) = 0 \\ &\mathbf{u} (\mathbf{x}', hl_r + 0; h) - \mathbf{u} (\mathbf{x}', hl_r - 0; h) = 0; \mathbf{x}' \in \omega, r \in [-m + 1, n - 1] \end{aligned} \tag{1.3}$$

are given on the layer boundaries.

Here $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement vector, $\boldsymbol{\sigma}^{(3)} = (\sigma_{13}, \sigma_{23}, \sigma_{33})$ is the stress vector acting over areas normal to the middle surface of the plate.

The boundary conditions on the faces and side surfaces $\partial\Omega$ of the slab Ω have the form

$$\boldsymbol{\sigma}^{(3)} (\mathbf{x}', hl_n - 0; h) = \mathbf{p}^+ (\mathbf{x}', h), \quad \boldsymbol{\sigma}^{(3)} (\mathbf{x}', hl_{-m}; h) = \mathbf{p}^- (\mathbf{x}', h); \quad \mathbf{x}' \in \omega \tag{1.4}$$

$$\mathbf{u} (\mathbf{x}, h) = 0; \quad \mathbf{x}' \in \partial\omega, \quad hl_{-m} \leqslant x_3 \leqslant hl_n \tag{1.5}$$

It is assumed that the boundary of the domain ω is a regular curve, there are no mass forces, the vectors of the right-hand sides in (1.4) and (1.5) can be represented as $\mathbf{p}^{\pm} (\mathbf{x}', h) = (h^{-1}\mathbf{p}, p_3)^{\pm}(\mathbf{x}')$ where $\mathbf{p} = (p_1, p_2)$, p_j $(j = 1, 2, 3)$ are smooth functions (of the class C^∞) of the coordinates x_1, x_2.

The asymptotic form of the solution of problem (1.1), (1.3) and (1.5) is represented in the form of power series in the parameter h. Outside a small neighbourhood of the plate edge the coefficients of these series are sought from the solutions $\mathbf{u}^{(k)}$ of problems in sections by a plane normal to its bases through the domain Ω, and the solutions of the limit problem in the domain ω. An algorithm for constructing the vectors $\mathbf{u}^{(k)}$ is described in Sect.2.1, while soltuions corresponding to the homogeneous problems, needed for consistency of the expansions on the limiting two-dimensional surface, are given in Sect.2.2.

The asymptotic representation $\mathbf{u}^h (\mathbf{x}, h)$ of the solution of the initial problem is presented in Sect.3 while the operator of the limit boundary value problem in the domain ω is constructed in Sect.4. The functions of the boundary layer orginating near the plate side surface are not examined in detail, only boundary conditions of the limit problem in $\partial\omega$ are formulated.

Furthermore, the physical meaning of the coefficients of the system of limit problem equations is refined in Sect.5, and examples of analysing the effective plate stiffness characteristics and a comparison of the results with those available in the literature are given in Sect.6.

2. **Auxiliary construction 2.1.** We consider problem (1.1) and (1.3) with the conditions (1.4) on the faces of the plate Ω and we seek its solution in the form of the series

$$\mathbf{V}(\mathbf{x}, h) = \sum_{k=0}^{\infty} h^k \mathbf{u}^{(k)}(\mathbf{x}', h^{-1}x_3) \tag{2.1}$$

Substitution of (2.1) into relationships (1.1) and comparison of the coefficients (after changing the coordinates $(\mathbf{x}', \mathbf{x}_3) \to (\mathbf{x}', \zeta) = (\mathbf{x}', h^{-1}\mathbf{x}_3)$) for identical powers of the parameter h results in a recursion sequence of problems on the segments $[l_{-m}, l_{-m+1}], \ldots, [l_{n-1}, l_n]$ of the $O\zeta$ axis for systems of ordinary differential equations with the parameter $\mathbf{x}' \in \omega$

$$\begin{gathered} B_3{}^r \frac{d^2}{d\zeta^2} \mathbf{u}^{(k)}(\mathbf{x}', \zeta) = -(L_1{}^r + L_1^{r,\,T})\left(\frac{\partial}{\partial \mathbf{x}'}\right) \frac{d}{d\zeta} \mathbf{u}^{(k-1)}(\mathbf{x}', \zeta) - \\ I^r\left(\frac{\partial}{\partial \mathbf{x}'}\right) \mathbf{u}^{(k-2)}(\mathbf{x}', \zeta) \\ I^r(\partial/\partial\mathbf{x}') = (K_1{}^T L_2{}^r + K_2{}^T L_1{}^r)(\partial/\partial\mathbf{x}'), \quad r \in [-m, n], \quad k = 0, 1, \ldots \end{gathered} \tag{2.2}$$

Matching conditions are derived analogously from relationships (1.3) for the coefficients $\mathbf{u}^{(k)}$ at the packet layer boundaries, as are the boundary conditions on its side surface, which have the form

$$\begin{gathered} B_3{}^r \frac{d}{d\zeta} \mathbf{u}^{(k)}(\mathbf{x}', l_r \mp 0) - B_3^{r\pm 1} \frac{d}{d\zeta} \mathbf{u}^{(k)}(\mathbf{x}', l_r \mp 0) = \\ L_1^{r\mp 1}\left(\frac{\partial}{\partial \mathbf{x}'}\right) \mathbf{u}^{(k-1)}(\mathbf{x}', l_r \pm 0) - L_1{}^r\left(\frac{\partial}{\partial \mathbf{x}'}\right) \mathbf{u}^{(k-1)}(\mathbf{x}', l_r \mp 0) + \\ (\delta_{k,0}(\mathbf{p}^- \delta_{r,-m} - \mathbf{p}^+ \delta_{r,n+1}), \delta_{k-1,0}(\mathbf{p}_3{}^- \delta_{r,-m} - \mathbf{p}_3{}^+ \delta_{r,n+1}))(\mathbf{x}') \end{gathered} \tag{2.3}$$

$$\mathbf{u}^{(k)}(\mathbf{x}', l_r \mp 0) - \mathbf{u}^{(k)}(\mathbf{x}', l_r \pm 0) = 0; \quad r \in [-m, n+1], \quad k = 0, 1, \ldots \tag{2.4}$$

The Hooke's law relationships are used in the form (1.2) in deriving (2.2)-(2.4) and henceforth. The notation $L_1 = B_2{}^T K_1 + B_3 K_2$, $L_2 = B_1 K_1 + B_2 K_2$ is still used here, where $K_j = K_j(\partial/\partial x')$ are matrix differential operators with the elements $k_{jj}{}^1 = k_{j3}{}^2 = k^1_{3(3-j)} = \partial/\partial x_j$; $j = 1,2$. It is assumed that functions with negative superscripts as well as layer characteristics with the numbers $r = 0$ and $r \notin [-m, n]$ equal zero. The plus and minus signs in (2.3) correspond to the superscripts $r \geqslant 0$ and $r < 0$, respectively. The boundary conditions on the plates faces are obtained from (2.3) for $r = -m$ and $r = n + 1$.

The homogeneous boundary value problem corresponding to (2.2)-(2.4) allows of the non-trivial solutions

$$\mathbf{v}_q^{(k)}(\mathbf{x}') = (\delta_{1,q}; \delta_{2,q}; \delta_{3,q}) \mathbf{v}_q^{(k)}(\mathbf{x}'); \quad q = 1, 2, 3; \quad k = 0, 1, \ldots \tag{2.5}$$

where $\mathbf{v}_q^{(k)}(\mathbf{x}')$ are arbitrary functions of the coordinates $x_1, x_2 \in \omega$. Consequently, the systems do not have solutions for arbitrary right-hand sides.

In order to construct the coefficients of the asymptotic representation (2.1), we consider the adjoint problem (2.2)-(2.4) in the domains

$$\Omega_+ = \{\mathbf{x} : \mathbf{x}' \in \omega, 0 \leqslant \zeta \leqslant l_n\}, \quad \Omega_- = \{\mathbf{x} : \mathbf{x}' \in \omega, l_{-m} \leqslant \zeta \leqslant 0\}$$

We shall consider the solvability conditions to be satisfied for these problems for fixed r for $r = n, \ldots, 1$ and $r = -m, \ldots, -1$. Then the coefficients $\mathbf{u}^{(k)}$ of **the representations** of the solution $\mathbf{V}(\mathbf{x}, h)$ in the domains $\Omega_\pm$ analogous to (2.1) can be found by evaluating known quadratures solving the problems (2.2)-(2.4) sequentially for $r \in [-m, n]$. Introducing in addition the normalization conditions $\mathbf{u}^{(k)}(\mathbf{x}', 0) = \mathbf{0}$ $(k = 0, 1, \ldots)$ we obtain for layers with the numbers $j \in [0, n]$

$$\begin{gathered} \mathbf{u}^{(k)}(\mathbf{x}', \zeta) = -K_2\left(\frac{\partial}{\partial \mathbf{x}'}\right) \int_0^\zeta \mathbf{u}^{(k-1)}(\mathbf{x}', t)\, dt + \\ \sum_{r=1}^{j} B_3^{r,-1} \Big\{ (l_r - l_{r-1}) \Big[\sum_{s=r}^{n} \int_{l_{s-1}}^{l_s} L^s(\mathbf{u}; k; \mathbf{x}', t)\, dt + \end{gathered} \tag{2.6}$$

$$(\delta_{k,0}\mathbf{p}^+, \delta_{k-1,0}p_3^+)(\mathbf{x}')\Big] - \int\limits_{l_{r-1}}^{l_r} (l_{r-1} - t)\, L^r(\mathbf{u}, k; \mathbf{x}', t)\, dt -$$

$$K^r\Big(\frac{\partial}{\partial \mathbf{x}'}\Big) \int\limits_{l_{r-1}}^{l_r} \mathbf{u}^{(k-1)}(\mathbf{x}', t)\, dt\Big\} -$$

$$B_3^{j,-1}\{(l_j - \zeta)\Big[\sum_{r=j}^{n} \int\limits_{l_{r-1}}^{l_r} L^r(\mathbf{u}; k; \mathbf{x}', t) dt + (\delta_{k,0}\mathbf{p}^+, \delta_{k-1,0}p_3^+)(\mathbf{x}')\Big] -$$

$$\int\limits_{\zeta}^{l_j} (\zeta - t)\, L^j(\mathbf{u}; k; \mathbf{x}', t)\, dt - K^{(j)}\Big(\frac{\partial}{\partial \mathbf{x}'}\Big) \int\limits_{\zeta}^{l_j} \mathbf{u}^{(k-1)}(\mathbf{x}', t)\, dt\}$$

$$L(\mathbf{u}; k; \mathbf{x}', t) = L_1{}^T \frac{d}{dt} \mathbf{u}^{(k-1)}(\mathbf{x}', t) + I\Big(\frac{\partial}{\partial \mathbf{x}'}\Big) \mathbf{u}^{(k-2)}(\mathbf{x}', t),$$

$$K\Big(\frac{\partial}{\partial \mathbf{x}'}\Big) = B_2{}^T K_1\Big(\frac{\partial}{\partial \mathbf{x}'}\Big)$$

Analogous formulas also hold for layers denoted by numbers in the interval $[-m, -1]$.

For the consistency of the coefficients constructed above for the asymptotic representations is it necessary still to satisfy the adjoint condition (2.3) in the section $\Pi = \{\mathbf{x} : \mathbf{x}' \in \omega,\ \zeta = 0\}$ of the plate Ω by the plane $\{\zeta = 0\}$. We take Π as a reduction surface, then the remaining relationships not taken into account in (2.3) will be satisfied if for $k = 0, 1, \ldots$ the expressions

$$\mathbf{F}^{(k)}(\mathbf{x}') = \sum_{r=1}^{n+1}\Big\{J_+{}^r\Big(\frac{\partial}{\partial \mathbf{x}'}\Big)\mathbf{u}^{(k-1)}(\mathbf{x}', l_{r-1}) + I^r\Big(\frac{\partial}{\partial \mathbf{x}'}\Big) \int\limits_{l_{r-1}}^{l_r} \mathbf{u}^{(k-2)}(\mathbf{x}', t)\, dt\Big\} + \tag{2.7}$$

$$\sum_{r=-m-1}^{-1} \Big\{J_-{}^r\Big(\frac{\partial}{\partial \mathbf{x}'}\Big)\mathbf{u}^{(k-1)}(\mathbf{x}', l_{r+1}) + I^r\Big(\frac{\partial}{\partial \mathbf{x}'}\Big) \int\limits_{l_{r+1}}^{l_r} \mathbf{u}^{(k-2)}(\mathbf{x}', t)\, dt\Big\} +$$

$$(\delta_{k,0}(\mathbf{p}^+ - \mathbf{p}^-), \delta_{k-1,0}(p_3{}^+ - p_3{}^-))(x')$$

$$\Big(J_\pm{}^r\Big(\frac{\partial}{\partial \mathbf{x}'}\Big) = \pm(L_1^{r\pm 1} - L_1{}^r)^T\Big(\frac{\partial}{\partial \mathbf{x}'}\Big)\Big)$$

are identically equal to zero for $\mathbf{x}' \in \omega$.

2.2. In the general case the right-hand sides of (2.7) are different from zero and conditions (2.3) are not satisfied on Π. To compensate for the residual $\mathbf{F}^{(k)}$ originating in each step in k, we construct the solution of the homogeneous problem (2.2)-(2.4).

$$\mathbf{U}^{(s)}(\mathbf{x}', \zeta; h) = \sum_{p=0}^{\infty} h^p \mathbf{v}^{(s,p)}(\mathbf{x}', \zeta) \tag{2.8}$$

The principal terms (in h) in (2.8) are the vector functions $\mathbf{v}^{(s,0)}(\mathbf{x}')$ given by (2.5). The boundary value problems to determine the coefficients of the series (2.8) agree with relationships (2.2)-(2.4) in which $\mathbf{p}^\pm = 0$, $p_3{}^\pm = 0$ while the subscript k is replaced by p.

As in Sect.2.1, let the **conditions for** problems (2.2)-(2.4) to be solvable be satisfied in the domains $\Omega_\pm$. Seeking the solutions of these problems successively in the layers $r = n, \ldots, 1$ and $r = -m, \ldots, -1$, we find the vectors

$$\mathbf{v}^{(s,1)}(\mathbf{x}', \zeta; r) = -\zeta(B_3^{r,-1}K^r + K_s)\Big(\frac{\partial}{\partial \mathbf{x}'}\Big)\mathbf{v}^{(s)}(\mathbf{x}') + \mathbf{v}_0^{(s)}(\mathbf{x}'; r) \tag{2.9}$$

in the first step (in h) of the algorithm for constructing the coefficients $u^{(k,p)}$.

Here $\mathbf{v}_0{}^{(s)}(\mathbf{x}'; r)$ are functions independent of ζ and determined **from the conditions of** continuity of the coefficients $\mathbf{v}^{(s,1)}$ on the connecting surfaces $\{\zeta = l_r\}$, $r = [-m+1, n-1]$ of the plate layers.

The vector-functions (2.9) also satisfy conditions (2.3) of continuity of the stress tensor components on the reduction surfaces. The other coefficients of series (2.8) are also determined in an analogous manner in the next steps of the algorithm in p: they are polynomials of the variable ζ of degree $p > 1$, but do not satisfy the conditions mentioned on Π. The vectors of the residuals originating in the right-hand sides of the **first** conditions (1.3) in Π generate the series

$$\sum_{p=0}^{\infty} h^{p-1} L^{(p)}\left(\frac{\partial}{\partial \mathbf{x}'}\right) \mathbf{v}^{(s)}(\mathbf{x}'); \quad s=0, 1, \ldots \tag{2.10}$$

when series (2.8) is substituted into the initial problem.

The elements of the matrix $L^{(p)}$ are homogeneous, of order p, differential expressions whose coefficients depend on the elastic constants and the height of the plate layers.

The constants of integration of the systems of differential equations (2.2) take part in the iteration processes when constructing the functions $\mathbf{v}^{(s,p)}(\mathbf{x}', \zeta; r)$. Let us eliminate this arbitrariness: we will assume that all the coefficients of series (2.8) vanish on the surface Π for $p > 0$, $k = 0, 1, \ldots$

3. **Asymptotic form of the solution.** The asymptotic representation (in h) $\mathbf{u}^h(\mathbf{x}, h)$ of the solution of the problem (1.1), (1.3)-(1.5) will be sought in the form (see /1, 5/)

$$\mathbf{u}^h(\mathbf{x}, h) = \mathbf{V}(\mathbf{x}'; \zeta; h) + \sum_{k, s=0}^{\infty} h^{k+s-1} \sum_{q=1}^{3} h^{-J_q} \mathbf{v}_q^{(k, s)}(\mathbf{x}', \zeta) + \mathbf{W}(\mathbf{x}, h) \tag{3.1}$$
$$(J_1 = J_2 = 1, \ J_3 = 2)$$

Here $\mathbf{V}(x', \zeta; h)$ is the solution (2.1) of problem (2.2)-(2.4), $\mathbf{v}^{(k,s)}(\mathbf{x}', \zeta)$ are coefficients of the expansion (2.8) of the solution of the corresponding homogeneous problem, and $\mathbf{W}(\mathbf{x}, h)$ is a function of the boundary layer originating near the side surface of the plate Ω.

Let the coefficients of the series $\mathbf{V}(\mathbf{x}', \zeta; h)$ and $\mathbf{U}^{(s)}(\mathbf{x}', \zeta; h)$ be found in the iteration processes of Sects.2.1 and 2.2. Then the first two components on the right-hand side of (3.1) satisfy (1.1), the second conditions of (1.3), conditions (1.4) and, for $r \neq 0$, and the first conditions of (1.3). In order for their sum to be an approximation to the exact solution $\mathbf{u}(\mathbf{x}, h)$ outside a small neighbourhood of the plates edge, additional conditions must be imposed on the coefficients of the representations (2.1) and (2.8).

Omitting the boundary layer function in (3.1), substituting the expression obtained in the initial problem (1.1)-(1.5) and taking account of (2.7) and (2.10), we find the residual vector in the first group of conditions (1.3) on the surface Π:

$$\mathbf{R}(\mathbf{x}') = \sum_{k=0}^{\infty} h^{k-1} \left\{ \mathbf{F}^{(k)}(\mathbf{x}') + \sum_{p=k}^{\infty} \sum_{q=1}^{3} h^{-1-J_q} L^{(p)}\left(\frac{\partial}{\partial \mathbf{x}'}\right) \mathbf{v}_q^{(p-k)}(\mathbf{x}') \right\} \tag{3.2}$$

We regroup the components in (3.2) so that each term of the series has an order in h no higher than k and depends on the coefficients $\mathbf{v}_q^{(p)}$ with superscipt $p \leqslant k$. Successively equating the expressions obtained to zero, we arrive at the relationships

$$\sum_{q=1}^{3} \sum_{p=1}^{3+J_q} h^{p-J_q-1} L^{(p)}\left(\frac{\partial}{\partial \mathbf{x}'}\right) \mathbf{v}_q^{(k)}(\mathbf{x}') = -\Phi^{(k)}(\mathbf{x}', h), \quad \mathbf{x}' \in \omega \tag{3.3}$$
$$\left(\Phi^{(k)}(\mathbf{x}', h) = \sum_{q=1}^{3} h^{J_q-1} \left(\sum_{p=1}^{k} L^{(p+J_q+1)}\left(\frac{\partial}{\partial \mathbf{x}'}\right) \mathbf{v}_q^{(k-p)}(\mathbf{x}') + \mathbf{F}^{(k+J_q-1)}(\mathbf{x}') \right)\right)$$

The left-hand sides of (3.3) contain the unknown functions $\mathbf{v}_q^{(k)}(\mathbf{x}')$ while the right-hand sides are evaluated in functions defined in Sect.2.1 and in coefficients $\mathbf{v}_q^{(p)}(\mathbf{x}')$ with superscripts $p < k$. The solutions $\mathbf{u}^{(k)}(\mathbf{x}', \zeta)$, $\mathbf{v}^{(s,p)}(\mathbf{x}', \zeta)$ of the adjoint problems considered in Sects.2.1 and 2.2 subject to the relationships (3.3), determine the asymptotic representation of a statically allowable displacement field for the initial problem (1.1)-(1.5) outside a small neighbourhood of the edge of the plate Ω.

The residuals introduced by the coefficients $\mathbf{u}^{(k)}$, $\mathbf{v}^{(s,p)}$ of the series (3.1) into conditions (1.5) on the plate side surface are compensated when constructing the boundary layer $\mathbf{W}(\mathbf{x}, h)$. The function $\mathbf{W}(\mathbf{x}, h)$ is represented in the form of a power series in the parameter h (see /7, 9/). The conditions for the exponential decrease of the expansion coefficients in each step in k enable us to find the values of the functions $\mathbf{v}_j^{(k)}(\mathbf{x}')$ $(j = 1, 2, 3)$ and the normal derivative of the function $\mathbf{v}_3^{(k)}(\mathbf{x}')$ on $\partial\omega$.

4. **Limit boundary value problem.** Relationships (3.3) and the conditions on $\partial\omega$ originating when constructing the boundary layer generate boundary value problems to determine the unknown functions $\mathbf{v}^{(k)}(\mathbf{x}')$ $(k = 0, 1, \ldots)$ in the domain ω.

The operator coefficients $L^{(p)}(\partial/\partial\mathbf{x}')$ in (3.3) are independent of the superscript k, consequently, the functions $\mathbf{v}^{(k)}(\mathbf{x}')$ satisfy the identical boundary value problem in the two-dimensional domain ω, which is indeed limiting with respect to the initial spatial problem (1.1)-(1.5). The operators $L^{(p)}(\partial/\partial\mathbf{x}')$ $(p = 0, 1, \ldots)$ are sought for a fixed s in each step p of the algorithm of Sect.2.2. The following matrices determine the left side of relationships (3.3):

$$L^{(0)}(\partial/\partial\mathbf{x}') = L^{(1)}(\partial/\partial\mathbf{x}') = 0,\ L^{(2)}(\partial/\partial\mathbf{x}') = K_1^T G^{(1)} K_1 (\partial/\partial\mathbf{x}') \tag{4.1}$$
$$L^{(3)}(\partial/\partial\mathbf{x}') = -K_1^T G^{(2)} K_1 K_2 (\partial/\partial\mathbf{x}') + K_2^T K_1^T G^{(2)} K_1 (\partial/\partial\mathbf{x}') + K_1^T H_1 K_1 (\partial/\partial\mathbf{x}')$$
$$L^{(4)}(\partial/\partial\mathbf{x}') = -K_2^T K_1^T G^{(3)} K_1 K_2 (\partial/\partial\mathbf{x}') + K_1^T H_2 K_2 (\partial/\partial\mathbf{x}')$$

Here $H_j(\partial/\partial\mathbf{x}')$ $(j = 1,2)$ are certain differential expressions, respectively, of the first and second orders with coefficients which depend on the height and the elastic constants of the layers comprising the plate. The matrices $G^{(i)}$ $(i = 1, 2, 3)$ have the form

$$G^{(i)} = \| g_{st}^{(i)} \|_{s,t=1}^{3} = \sum_{r=1}^{n+1} \int_{l_{r-1}}^{l_r} t^{i-1} \mathrm{B}^r\, dt + (-1)^{i+1} \sum_{r=-m}^{-1} \int_{l_r}^{l_{r-1}} t^{i-1} \mathrm{B}^r\, dt \tag{4.2}$$
$$(B = A_1^{-1} = B_1 - B_2 B_3^{-1} B_2^T)$$

where $\mathbf{A}$ is the matrix of the elastic compliances associated with the matrix $\mathbf{B}$ in (1.2).

System (3.3) of the limit problem is represented in the form

$$L\left(\frac{\partial}{\partial\mathbf{x}'}\right)\mathbf{v}^{(k)}(\mathbf{x}') = \Psi^{(k)}(\mathbf{x}'),\quad \mathbf{x}' \in \omega \tag{4.3}$$
$$\Psi^{(k)}(\mathbf{x}') = \left(-\Phi^{(k)} - K_1^T \sum_{j=1}^{2} H_j K_j \mathbf{v}^{(k-1)}, \Phi_3^{(k+1)}\right)(\mathbf{x}'),\quad \Phi = (\Phi_1, \Phi_2)$$

The operator $L(\partial/\partial\mathbf{x}')$ has the form of a symmetric matrix whose rows are the expressions

$$(E_{11}, E_{12}, -S_1)\left(\frac{\partial}{\partial\mathbf{x}'}\right),\quad (E_{12}, E_{22}, -S_2)\left(\frac{\partial}{\partial\mathbf{x}'}\right), \tag{4.4}$$
$$(-S_1, -S_2, D)\left(\frac{\partial}{\partial\mathbf{x}'}\right)$$

It follows from (4.1), (4.3) and the definitions of the operators $K_j(\partial/\partial\mathbf{x}')$ in Sect.2 that the operator $\mathbf{E}(\partial/\partial x') = \| E_{ij} \|_{i,j=1}^{2}(\partial/\partial\mathbf{x}')$ is **identical** with the unique (2×2)-matrices $L^{(2)}$, by a minor **different from zero**, while the operator $\mathbf{D}(\partial/\partial\mathbf{x}')$ is identical, respectively, with the element d_{33} of the matrix $K_2^T K_1^T G^{(3)} K_1 K_2$. The vector $(S_1, S_2, 0)^T$ is the third column of the matrix $K_1^T G^{(2)} K_1 K_2$ or, equivalently, is obtained by transposition of the third row of the matrix $K_2^T K_1^T G^{(2)} K_1$.

The constructions carried out here show that the operator $L(\partial/\partial\mathbf{x}')$ of the limiting boundary value problem is selfadjoint and satisfies the condition of strong ellipticity. The boundary of the domain ω is smooth and the systems $(\mathbf{v}^{(k)}, \partial v_3^{(k)}/\partial\mathbf{n})(\mathbf{x}')$, $\mathbf{x}' \in \partial\omega$ are the missing boundary condtions for (4.3). Consequently, problems (4.3) to determine the functions $\mathbf{v}^{(k)}(\mathbf{x}')$ are solvable single-**valuedly** and, therefore, the description of the algorithm for constructing the coefficients of the representation (3.1) is completed.

5. Effective stiffness characteristics of a laminar plate.

The functions $v_j^{(0)}(\mathbf{x}')$ $(j = 1, 2, 3)$ determine the principal term (in h) of the asymptotic representation (3.1) of the exact solution $\mathbf{u}(\mathbf{x}, h)$ of problem (1.1)-(1.5), while the expressions $\mathbf{v}(\mathbf{x}', h) = h^{-2}\mathbf{v}^{(0)}(\mathbf{x}') = h^{-2}(v_1, v_2)^{(0)}(\mathbf{x}')$ and $W(\mathbf{x}', h) = h^{-3} v_3^{(0)}(\mathbf{x}')$ are, respectively, the highest tangential displacements in the expansions in h, and the deflection of the reduction surface Π. The vector-function $(\mathbf{v}, w)(\mathbf{x}', h)$ satisfies system (4.3) in ω. **After** obvious substitutions, expressions (4.4) take the form

$$(hE_{11}, hE_{12}, -h^2 S_1)\left(\frac{\partial}{\partial\mathbf{x}'}\right),\quad (hE_{12}, hE_{22}, -h^2 S_2)\left(\frac{\partial}{\partial\mathbf{x}'}\right) \tag{5.1}$$
$$(-h^2 S_1, -h^2 S_2, h^3 D)\left(\frac{\partial}{\partial\mathbf{x}'}\right)$$

which is in agreement with the results obtained in /5/ by another method.

Let us consider the matrices $h^i G^{(i)}$ $(i = 1, 2, 3)$, defined by (4.2) and occurring in the definition of the operator of the limit problem (4.3). Their elements are coefficients of the differential expressions (5.1) and are the effective stiffness characteristics of the plate Ω in tension-shear and bending-torsion, and also describe the cross effects of interaction of the mentioned states of the packet. The matrices $hG^{(1)}$, $h^2G^{(2)}$, $h^3G^{(3)}$ are analogues of the static moments used in classical theory for a section of a plate of the zero-th, first, and second orders, respectively.

Let d_r be the height and t_r the ordinate of the r-th layer middle surface. Using representation (4.2) we find

$$hG^{(1)} = \sum_{r=-m}^{n} B^r d_r, \quad h^2G^{(2)} = \sum_{r=-m}^{n} B^r d_r t_r, \tag{5.2}$$

$$h^3G^{(3)} = \sum_{r=-m}^{r=n} B^r d_r \left(\frac{d_r^{\,2}}{12} + t_r^{\,2}\right)$$

It follows from (5.2), in particular, that the elements S_j, $j = 1, 2$ of the operator $L(\partial/\partial\mathbf{x}')$ equal zero for a plate of symmetric construction. Examples will be examined in Sect. 6 that show that analogous equations can describe the state of stress and strain of plates of non-symmetric construction.

6. Examples. 1°. Let the materials of the packet layers be isotropic and let ν_r and E_r be Poisson's ratio and Young's modulus of the r-th layer; $r \in [-m, n]$. The elements $\| \beta_{ij}{}^r \|_{i,j=1}^{3}$ of the matrix B^r in (4.2) take the form

$$\beta_{11}{}^r = \beta_{22}{}^r = \beta^r = E_r(1 - \nu_r^{\,2})^{-1}, \quad \beta_{12}{}^r = \beta_{21}{}^r = \nu_r\beta^r$$

$$\beta_{33}{}^r = 2^{-1}(1 - \nu_r)\beta^r, \quad \beta_{23}{}^r = \beta_{32}{}^r = 0$$

The neutral surface of the packet Ω is a distance

$$z_0 = \Sigma\beta^r d_r t_r (\Sigma\beta^r d_r)^{-1}$$

from the reduction plane Π.

Here and henceforth, summation is with respect to r and s between $-m$ and n. Passage to it in (3.1) is realized by renormalization of the coefficients $\mathbf{v}^{(k,p)}(\mathbf{x}', \xi; r)$ by the conditions

$$\mathbf{v}^{(k,p)}(\mathbf{x}', h^{-1}z_0; r) = 0; \quad k = 0, 1, \ldots; \quad p = 1, 2, \ldots \tag{6.1}$$

The operator $L(d/d\mathbf{x}')$ takes the form of a block matrix: the tangential displacements $\mathbf{v}^{(k)}(\mathbf{x}')$ of the neutral surface satisfy the system of equations of the generalized plane state of stress, while the deflection $w^{(k)}(\mathbf{x}')$ satisfies the Sophie Germain equation. Taking (6.1) into account, representations for the effective elastic characteristics of a plate are derived directly from (5.1)

$$\nu_* = \Sigma\beta^r\nu_r d_r(\Sigma\beta^r d_r)^{-1}$$

$$E_* = \Sigma\beta^r E_r^{-1}(1 - \nu_r)\,d_r(\Sigma\beta^r d_r)^{-1}\Sigma\beta^r(1 + \nu_r)\,dr$$

The effective cylindrical stiffness of the plate Ω

$$D_* = \frac{1}{12}\Sigma\beta^r d_r^{\,3} + \Sigma\beta^r\beta^s d_r d_s t_r(t_r - t_s)(\Sigma\beta^r d_r)^{-1}$$

is made up of the stiffness $D^r = \beta^r d_r^{\,3}/12$ of each layer and the apparent stiffness of the packet due to layer interaction. For the special case of a plate of symmetric construction ($d_r = d_{-r}$, $\beta^r = \beta^{-r}$, $t_r = t_{-r}$) the formulas presented agree with the results obtained in /13/. Comparing D_* with the cylindrical stiffness of a packet formed by alternating soft and hard layers /14/ we see that the corresponding apparent stiffnesses can differ substantially depending on the relationships between the elastic and geometric parameters of the layers.

2°. Now let the packet consist of orthotropic layers with identical elastic moduli E_1, E_2, ν_{12}, ν_{21} one of the principal elasticity axes of each layer agrees with the Ox_3 axis in direction while the other two make the angles γ_r, $r \in [-m, n]$ with the Ox_1 and Ox_2 axes. To investigate the effective elastic properties of the packet it is natural to introduce a coordinate system rotated through a certain angle $\varphi > 0$ in the x_1Ox_2 plane.

The matrices B^r are computed by formulas to transform the elastic constants during rotation of the coordinate axes /13/. In this case, unlike Example 1°, the elements $\beta_{13}{}^r$ and $\beta_{23}{}^r$ of the matrix B^r do not equal zero and have the following form in the above-mentioned coordinate system:

$$\beta_{13}{}^r = \sin 2(\gamma_r - \varphi)\left[\left(E_2 + \frac{E_1\nu_{21}}{2(1 - \nu_{12}\nu_{21})}\right)\cos 2(\gamma_r - \varphi) + \right.$$
$$\left.\frac{1}{2(1 - \nu_{12}\nu_{21})}(E_2\sin^2(\gamma_r - \varphi) - E_1\cos^2(\gamma_r - \varphi))\right]$$

$$\beta_{23}{}^r = \sin 2(\gamma_r - \varphi)\left[-\left(E_1 + \frac{E_2\nu_{12}}{2(1 - \nu_{12}\nu_{21})}\right)\cos 2(\gamma_r - \varphi) + \right.$$
$$\left.\frac{1}{2(1 - \nu_{12}\nu_{21})}(E_2\cos^2(\gamma_r - \varphi) - E_1\sin^2(\gamma_r - \varphi))\right]$$

The plate neutral surface coincides with the middle plane of the packet. The operator $L(d/dx')$ has the form of a block matrix here, as in Example 1°.

The expressions presented above and (5.1) show that the tangential displacements of the packet neutral surface satisfy the equations of the generalized **plane** state of stress of an orthotropic body for a certain fixed value φ_* of the angle φ if the following condition is satisfied:

$$\Sigma d_r \sin 4\gamma_r \Sigma d_r \cos(2\gamma_r - \pi/4) \Sigma d_r \sin(2\gamma_r - \pi/4) + \Sigma d_r \sin 2\gamma_r \Sigma d_r \cos 2\gamma_r \Sigma d_r \cos 4\gamma_r = 0 \tag{6.2}$$

Let the relationships (6.2) be satisfied. Then the desired angle is the solution of the equation

$$\operatorname{tg} 2\varphi_* = \Sigma d_r \sin 2\gamma_r (\Sigma d_r \cos 2\gamma_r)^{-1}$$

By calculating the coefficients of the differential operator $D(d/dx')$ in (5.1) analogously. it can be shown that the deflection of the neutral surface satisfies the orthotropic plate bending equation if (6.2) holds, in which the expression $D_r = d_r^3/12 + d_r t_r (t_r - l/2)$ has replaced d_r, where l is the packet height. The principal elasticity axes of a laminar plate here turn out to be rotated through the angle

$$\varphi_{**} = \frac{1}{2} \operatorname{arctg} [\Sigma D_r \sin 2\gamma_r (\Sigma D_r \cos 2\gamma_r)^{-1}] \tag{6.3}$$

It follows from (4.2) and (4.3) that if the principal elasticity axes (of the same or different kinds) of the layers are in agreement then a packet of non-symmetric construction will behave like a plate fabricated from a homogeneous orthotropic material.

The effective elastic constants $E_1^*, E_2^*, \nu_{12}^*, \nu_{21}^*, G^*$ are computed exactly as in example 1°. For the special case of a plate of regular construction $(d_r = d, \ \gamma_r = \pi r n^{-1}, \ r \in [-n, n])$ the state of stress and strain is characterized by the isotropy of the elastic properties noted in /12/ under tension or shear. It follows directly from (5.2) that under bending the function describing the deflection of the neutral surface of a packet of the structure mentioned will satisfy the orthotropic plate bending equation whose principal elasticity axes make the angles defined by (6.3) with the coordinate axes Ox_1 and Ox_2.

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